Original article
UDC 524.882
DOI: https://doi.org/10.18127/j20700970-202403-06

Reissner–Nordström Black Hole: Curvature and Singularity with Quantized Fundamental Tensor

A.N. Tawfik[1], **A.A. Alshehri**[2]

[1] Future University in Egypt (FUE), Fifth Settlement (11835 New Cairo, Arab Republic of Egypt)
[2] University of Hafr Al Batin (Hafar Al Batin 39524, Kingdom of Saudi Arabia)
[1] a.tawfik@fue.edu.eg

Abstract

To reveal the nature of curvatures and singularitis which are emerged with the proposed quantization imposed on the fundamental tensor, the timelike geodesic congruence of the Reissner-Nordström metric shall be derived, analytically, and analyzed, numerically. The evolution of the geodesic congruence expansion is found nonvanishing everywhere. Furthermore, as the radial distance from the singularity decreases, an extremely large geodesic congruence expansion evolution occurs. The proposed quantization seems to largely enhance and apparently enrich the profile of the geodesic congruence expansion evolution. That the Kretschmann scalar for both versions of the fundamental tensor is found finite everywhere allows for an unambiguous assessment that the curvatures and singularities are likely real and essential (not artifact in some coordinate systems). We conclude that the proposed quantization seems to locally sharpen the curvatures and hence the singularities of the charged, non-rotating, spherically symmetric, and massive Reissner-Nordström black hole. This finding would alter the Schwarzschild radius and even the entire black hole geometry, especially at relativistic quantum scales. We also conclude that the additional curvatures even with their approximate qualitative estimation point to a rich spacetime structure which is apparently overseen in the classical limit.

Keywords

Space and initial singularities; Invariant scalars in general relativity; Discretized curved spacetime; Riemann-Finsler-Hamilton geometry; Noncommutative differential geometry

For citation

Tawfik A.N., Alshehri A.A. Reissner-Nordström Black Hole: Curvature and Singularity with Quantized Fundamental Tensor. Nonlinear World. 2024. V. 22. № 3. P. 49–61. DOI: https://doi.org/10.18127/ j20700970-202403-06

A brief version in Russian is given at the end of the article

Introduction

Probing the spacetime at relativistic energy reveals that the continuous, precise, and smooth differential texture should be fine-structured. Then, its foaming, nonlocalizing, discretizing, and fluctuating nature becomes dominant [1]. Indeed, on Finsler manifold with the phase-space dimensions (coordinates and momenta), the quantum geometry, which uses differential geometric ingredients of the symplectic manifolds or phase-spaces to define a quantum theory, unveils genuine curvatures, connections, and geometric structures [2-5]. The theory proposed for dynamics of a quantum particle with positive mass m moving in the Finsler space establishes a framework for the proper inclusion of quantum-mechanical ingredients in the fundamental tensor. The Finsler structure shall be extended to auxiliary four-vectors of coordinates x_0^α and momenta p_0^β The relativistic generalized uncertainty principle (RGUP), which accounts for the consequences of the relativistic energies and gravitational fields in the fundamental theory of quantum mechanics (QM), the Heisenberg uncertainty principle, can be applied, directly. That is the generalization of the momentum operator $\hat{p}_0^\nu$ to $\varphi\hat{p}_0^\nu$, where the parameter φ is given as $\varphi = 1+\beta p_0^\rho p_{0\rho}$, where β is the RGUP parameter whose upper bound can be set, empirically and/or observationally [6]. The Finsler metric can be straightforwardly deduced from the Hessian of the resulting Finsler structure $F^2\left(x_0^\alpha, \varphi p_0^\beta\right)$. By parameterizing the coordinates on Finsler manifold and then equating the measures of the line element on both Finsler and Riemann manifolds, the quantized fundamental tensor in Riemann space, four dimensional, could be determined [5, 7, 8].

With RGUP, quantum geometry, and dynamics of a quantum particle in the phase-space dimensional Finsler geometry [2–5], the basic premise behind general relativity is intentionally preserved. All assumptions of general relativity are presumably maintained. The resulting quantized fundamental tensor $\tilde{g}_{\alpha\beta}$, which looks as a conformal transformation of the conventional $g_{\alpha\beta}$ [5, 7–9], seems to preserve the symmetric property and even to share with Weyl tensor the invariance property under the conformal change to $g_{\alpha\beta}$. On the other hand, with $\tilde{g}_{\alpha\beta}$ the applicability of GR is likely extended so that thereby the GR's sensical predictions are enlarged to cover the relativistic and quantum regimes as well.

The present study resumes the timelike geodesic congruence in homogeneous, isotropic, and spherically symmetric cosmic background, including Schwarzschild, de Sitter-Schwarzschild, and Friedmann-Lemaître-Robertson-Walker (FLRW) metrics [10]. This time, the curvature and singularity of Reissner-Nordström black hole shall be analyzed with $\tilde{g}_{\alpha\beta}$ and compared with the conventional $g_{\alpha\beta}$. The additional curvatures and the corresponding singularities which are emerged in the spacetime due to the proposed quantization of $g_{\alpha\beta}$ shall be determined by analyzing the evolution of a family of trajectories, where the trajectory congruence is described by flow lines generated by velocity fields [11, 12]. To assess whether the additional curvatures and their singularities are real and intrinsic, i.e., to diagnose any possible artifacts in some coordinate systems, Kretschmann scalar shall be utilized.

The present manuscript is organized as follows. The formalism is introduced in Section II. The geodesic equations with $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ shall be outlined in Section II A. Section II B1 reviews the timelike geodesic congruence with $g_{\alpha\beta}$, Section II B1, and $\tilde{g}_{\alpha\beta}$, Section II B2. The Kretschmann scalar shall be formulated in Section II C. Section III discusses the numerical analysis, where the timelike geodesic congruence with $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ shall be presented in Section III A and the Kretschmann scalar in Section III B. Section IV is devoted to the conclusions.

II. Formalism

All information about the quantized spacetime is conjectured to be encoded in the quantized fundamental tensor [2–5]

$$\tilde{g}_{\alpha\beta} = C g_{\alpha\beta}.$$

The quantity C stores the quantum-mechanically induced revision of $g_{\alpha\beta}$ [2–5]. An exact expression for C still poses a tough mathematical task. Until resolving this challenge, the approximation in Eq. (1) seems to remain unavoidable. To draw a picture about the necessity of such approximation, let us recall that the quantized fundamental tensor was estimated as [2-5]

$$\begin{aligned}\tilde{g}_{\mu\nu} &= \left(\varphi^2 + 2\frac{\kappa}{\left(p_0^0\right)^2}F^2\right)\left[1+\frac{m^2}{f^2}\left(1+2\beta p_0^\rho p_{0\rho}\right)\dot{p}_0^\mu \dot{p}_0^\nu\right]g_{\mu\nu} + \\ &+\left[\frac{dx_0^\mu}{d\zeta^\mu}\frac{dx_0^\nu}{d\zeta^\nu} + \bar{m}^2\left(1+2\beta p_0^\rho p_{0\rho}\right)\frac{dp_0^\mu}{d\zeta^\mu}\frac{dp_0^\nu}{d\zeta^\nu}\right]d_{\mu\nu},\end{aligned} \tag{1}$$

where $\bar{m}$ is the mass of the quantum particle normalized to the Planck mass. f represents the maximal proper gravitational force under which the quantum particle shall be gravitationally maximally accelerated along its motion in the additional curvature. f was suggested as a new physical constant [7]. F belongs to the simplest Finsler metrics, i.e., Klein metric. ζ^μ are parametrizations relating the coordinates in the Finslerian tangent bundle to the Riemann coordinates. The approximation that C is just given as

$$C=\left(\varphi^2+2\frac{\kappa}{\left(p_0^0\right)^2}F^2\right)\left[1+\frac{\bar{m}^2}{f^2}\left(1+2\beta p_0^\rho p_{0\rho}\right)\dot{p}_0^\mu \dot{p}_0^\nu\right], \tag{3}$$

leads to Eq. (1), on one hand. On the other hand, it means that the second line of Eq. (2) vanishes, especially the discretized Finsler metric

$$d_{\mu\nu}=2\frac{\kappa}{\left(p_0^0\right)^2}\left\{4\varphi F^2 l_\mu l^\sigma g_{\sigma\mu}-4\varphi F^3 l^\sigma\left[\delta_{0\nu}g_{\sigma\mu}+\delta_{0\mu}g_{\sigma\nu}\right]+F^4\left(2+\varphi\right)\delta_{0\mu}\delta_0^\sigma g_{\sigma\nu}\right\}. \tag{4}$$

There is a rigorous mathematical restriction disfavoring this approximation. This restriction is set by the parameter φ, which does not depend either on x_0 or $d_{\mu\nu}$. Therefore, φ must remain finite and consequently $d_{\mu\nu}$. Nonvanishing $d_{\mu\nu}$ makes it almost impossible to equate the Finslerian and Riemannian measures of the line element. The truncation of Eq. (2) allows at least a partial preservation of the quantum-mechanical revision. On the other hand, the present qualitative study is not exclusively impacted by a precise quantitative estimation of the quantization outlined in Eq. (2). In this regard, we emphasize that Eq. (3) introduces an approximate generalization to (1), where

$$\varphi=1+2\beta p_0^\rho p_{0\rho}=1+\frac{\kappa}{\left(p_0^0\right)^2}F^2, \tag{5}$$

$$\varphi_\mu=\frac{2\kappa F}{\left(p_0^0\right)^3}\left(p_0^0 l_\mu-F\delta_{0\mu}\right), \tag{6}$$

$$\varphi_{\mu\nu}=\frac{2\kappa}{\left(p_0^0\right)^2}g_{\mu\nu}(x)-\frac{4\kappa F}{\left(p_0^0\right)^3}\left(l_\nu\delta_{0\mu}+l_\mu\delta_{0\nu}\right)+\frac{6\kappa F^2}{\left(p_0^0\right)^4}\delta_{0\nu}\delta_{0\mu}, \tag{7}$$

$$F^2=\varphi^2\frac{\left|p_0^\nu\right|^2-\left|x_0^\mu\right|^2\left|p_0^\nu\right|^2+\left\langle x_0^\mu\cdot p_0^\nu\right\rangle^2}{1-\left|x_0^\mu\right|^2}, \tag{8}$$

$$l_\gamma=\frac{p_0^\gamma}{F}+\frac{x_0,p_0}{\left(1-\left|x_0\right|^2\right)F}x_0^\gamma. \tag{9}$$

It is correct that the proposed approximation reduces the precision of the analytical derivation. On the other hand, the present numerical analysis considers the mean values of the quantities composing C, so that C is treated as a correction to $g_{\alpha\beta}$, which in turn means that $g_{\alpha\beta}$ is fully retained at $C=1$. Whatever values C takes, the final conclusions are qualitatively preserved and seem to manifest remarkable insights on the quantization of spacetime. To avoid controversy, the present calculations set C as a free parameter. Thereby, we seek out at which value of C, the proposed quantization emerges. On the other hand, the present script does not focus on the fundamental tensor and its quantization. It rather deals with the additional curvatures in which additional connections and geometric structures emerge. This might explain the strong dependence of the timelike geodesic congruence on C, Fig. 1.

A. Geodesic Equations

The linear dependence of $\tilde{g}_{\alpha\beta}$ on $g_{\alpha\beta}$, Eq. (1), suggests that the proper time $-c^2 d\tau^2=ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ is obviously valid for the quantization of $g_{\alpha\beta}$, so that in natural units, we get $-d\tilde{\tau}^2=d\tilde{s}^2=\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta$ and

$$\tilde{\tau}_{ab} = \int_0^1 \sqrt{-\tilde{g}_{\alpha\beta}(x)\frac{dx^\alpha}{d\zeta}\frac{dx^\beta}{d\zeta}} = \int_0^1 L\left(\frac{dx^\alpha}{d\zeta}, x^\alpha\right) d\zeta. \tag{10}$$

With Euler-Lagrange equations,

$$\frac{-d}{d\zeta}\frac{\partial L}{\partial\left(\frac{dx^\gamma}{d\zeta}\right)} + \frac{\partial L}{\partial x^\gamma} = 0, \tag{11}$$

is obtained, where L is the Lagrangian. Then, the quantized geodesic equations becomes

$$\frac{d^2 x^\alpha}{d\tau^2} + \tilde{\Gamma}^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau} = \frac{-g_{\delta\beta}}{2\tilde{g}_{\alpha\gamma}} F^2_{,\gamma} C \frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}, \tag{12}$$

We remark that the quantization comes up with an additional term, on right hand site. Also, the quantized affine connections adds another term [2–5],

$$\tilde{\Gamma}^\alpha_{\delta\beta} = \Gamma^\alpha_{\delta\beta} + \frac{F^2_{,\gamma}}{2C}\left(\delta^\alpha_\beta + \delta^\alpha_\delta - g^{\alpha\gamma} g_{\delta\beta}\right), \tag{13}$$

where the derivative of the Klein metric F reads

$$F^2_{,\gamma} = 2\frac{\left\langle x_0^\alpha \cdot p_0^\beta \right\rangle}{\left[\left(x_0^\alpha\right)^2 - 1\right]^2}\left\{\left[1 - \left(x_0^\alpha\right)^2\right] p_0^\beta - x_0^\alpha \left\langle x_0^\alpha \cdot p_0^\beta \right\rangle\right\}. \tag{14}$$

At the end of the day, the present manuscript suggests an analytical and numerical estimation of the additional contributions to the timelike geodesic congruence of the Reissner-Nordström black-hole. Concretely, the present study compares between conventional and quantized timelike geodesic congruence.

Section II B1 introduces the analytical expressions for the timelike geodesic congruence with $g_{\alpha\beta}$ [10]. The corresponding expressions with $\tilde{g}_{\alpha\beta}$ shall be elaborated in section II B2.

B. Timelike Geodesic Congruence

In the gravitational field of a charged, non-rotating, spherically symmetric, and massive Reissner-Nordström black hole [13–16], the static solution of the Einstein-Maxwell field equations refers to the line element

$$ds^2 = -\Delta(r)\,dt^2 + \frac{1}{\Delta(r)}dr^2 + r^2 d\Omega^2, \tag{15}$$

where r is the radial distance and t is the time coordinate whose measure is made by a stationary clock positioned at infinity. $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the standard line element on the surface of two-sphere with (θ, φ) are the spherical angles. $\Delta(r) = 1 - 2M/r + Q^2/4\pi\varepsilon r^2$, where $\varepsilon = \left(\mu_0 c^2\right)^{-1}$ is the electric constant, c is the speed of light and μ_0 is the vacuum permeability or magnetic constant. At mass M and electric charge Q, $\Delta(r)$ can be expressed as

$$\Delta(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}, \tag{16}$$

with the Schwarzschild radius $r_s = 2M$ and $r_Q^2 = Q^2/4\pi\varepsilon$. The timelike geodesic congruence with conventional and quantized fundamental tensor shall be derived in section II B1 and II B2, respectively.

1. Congruence Conventional fundamental tensor $g_{\alpha\beta}$

In comoving coordinates, $x^\alpha = (t, r, \theta, \varphi)$, the first geodesic equation reads

$$\frac{du^t}{d\tau} = \frac{2\left(Q^2 - 4\pi M\varepsilon r\right)u^t}{r\left[Q^2 + 4\pi\varepsilon r(r-2M)\right]}\frac{dr}{d\tau}, \tag{17}$$

which can be solved as

$$u^t = \frac{r^2}{Q^2 + 4\pi\varepsilon r(r-2M)}. \tag{18}$$

At vanishing third and fourth components of the velocity field, that are u^θ and u^φ, and under the normalization condition $u^t u^r g_{tr} = -1$, the second geodesic equation becomes

$$\frac{du^r}{d\tau} = \frac{1}{2}\Delta(r)\left[\frac{(Q^2 - 4\pi\varepsilon Mr)}{2\pi\varepsilon r^3}\left(u^t\right)^2 - \frac{8\pi\varepsilon r(Q^2 + 4\pi\varepsilon Mr)}{\left[Q^2 + 4\pi\varepsilon r(r-2M)\right]^2}\left(u^r\right)^2 + 2r\left[\left(u^\theta\right)^2 + \sin^2\theta\left(u^\varphi\right)^2\right]\right], \tag{19}$$

and can be solved for u^r,

$$u^r = \frac{1}{4}\left[\frac{r^2 - 4\pi\varepsilon Q^2 - 16\pi\varepsilon^2 r(r-2M)}{\pi\varepsilon\left[Q^2 + 4\pi\varepsilon r(r-2M)\right]}\right]^{\frac{1}{2}}\left[4 + \frac{Q^2 - 8\pi\varepsilon Mr}{\pi\varepsilon r^2}\right]^{\frac{1}{2}}. \tag{20}$$

With these two finite velocity fields, the geodesic congruence expansion Θ can be derived as

$$\Theta = u^\alpha_{;\alpha} = u^\alpha_{,\alpha} + u^\sigma \Gamma^\alpha_{\sigma\alpha} = \frac{r^2 - 2\pi\varepsilon Q^2 + 8\pi\varepsilon^2 r(3M - 2r)}{2\pi\varepsilon^{\frac{3}{2}} r^2\left[Q^2 + 4\pi\varepsilon(r-2M)\right]^{\frac{1}{2}}\left[\frac{r^2 - 4\pi\varepsilon Q^2 - 16\pi\varepsilon^2 r(r-2M)}{\pi\varepsilon\left[Q^2 + 4\pi\varepsilon(r-2M)\right]}\right]^{\frac{1}{2}}}. \tag{21}$$

Then, the evolution of the geodesic congruence expansion, Eq. (21), reads

$$\frac{d\Theta}{d\tau} = \frac{-1}{8\pi\varepsilon^2 r^4\left[r^2 - 4\pi\varepsilon Q^2 - 16\pi\varepsilon^2 r(r-2M)\right]}\times$$
$$\times\left\{r^4 - 6\pi\varepsilon Q^2 r^2 + 16\pi\varepsilon^2\left(Q^4 + 4Mr^3 - 2r^4\right) + 32\pi\varepsilon^3 Q^2 r(3r - 8M) + 128\pi\varepsilon^4 r^2\left(9M^2 - 8Mr + 2r^2\right)\right\}. \tag{22}$$

2. Quantized fundamental tensor $\tilde{g}_{\alpha\beta}$

In a similar manner, in comoving coordinates, the velocity fields $\tilde{u}^t$ and $\tilde{u}^r$ can be derived for quantized fundamental tensor, affine connections, and geodesic equations

$$\tilde{u}^t = \exp\left[2\ln(r) - \ln(\Delta(r)) - \frac{r}{C}F^2_{,\gamma}\right], \tag{23}$$

$$\tilde{u}^r = \frac{1}{r}\left[r^2\exp\left(\frac{-2rF^2_{,\gamma}}{C}\right) - \frac{\Delta(r)}{C}\right]^{\frac{1}{2}}. \tag{24}$$

The quantized geodesic congruence expansion becomes

$$\tilde{\Theta} = \tilde{u}^{\alpha}_{;\alpha} = \tilde{u}^{\alpha}_{,\alpha} + \tilde{u}^{\sigma}\tilde{\Gamma}^{\alpha}_{\sigma\alpha} = \frac{\exp\left(-\frac{2rF^2_{,\gamma}}{C}\right)}{2C^{\frac{3}{2}}r^2\Delta(r)}\left[Cr^2\exp\left(-\frac{2rF^2_{,\gamma}}{C}\right)-\Delta(r)\right]^{-1/2}$$

$$\left\{2Cr^3\left[\Delta(r)F^2_{,\gamma}+C\frac{\partial\Delta(r)}{\partial r}\right]+\Delta(r)\exp\left(\frac{2rF^2_{,\gamma}}{C}\right)\left[2\Delta(r)\left(C-2rF^2_{,\gamma}\right)-3Cr\frac{\partial\Delta(r)}{\partial r}\right]\right\}. \quad (25)$$

Then, the evolution of $\tilde{\Theta}$ reads

$$\frac{d\tilde{\Theta}}{d\tau} = \frac{\exp\left(\frac{-2rF^2_{,\gamma}}{C}\right)}{4C^2r^4\Delta^2(r)\left[\Delta(r)\exp\left(\frac{2rF^2_{,\gamma}}{C}\right)-Cr^2\right]}$$

The numerical analysis of $d\Theta/d\tau$, Eq. (22), and $d\tilde{\Theta}/d\tau$, Eq. (26), shall be discussed in Section III A. To assess whether the additional curvatures and corresponding singularity are essential and intrinsic or artifact in some coordinate systems, the quadratic invariant Kretschmann scalar shall be utilized, Section II C.

C. Nature of emerged Curvature and Singularity

One would rightfully argue that the curvature emerged with the proposed quantization of the fundamental tensor, $\tilde{g}_{\alpha\beta}$, as well as the associated singularity might be artifacts in some coordinate systems. For instance, the additional curvatures, connections, and geometric structures might emerge only in the spherical coordinates which are assumed in the present study. The procedure to discover whether the curvature and singularity are real and essential is their invariance in all coordinate transformations.

An assessment whether the curvature and singularity are artifact or intrinsic in all coordinate systems is only feasible by the invariant scalars of general relativity. While the Ricci scalar in the vacuum solutions is vanishing everywhere, the Kretschmann scalar can be vanishing (artifact and removable) or finite (real and intrinsic curvatures). The Kretschmann scalar belongs to the basic polynomial curvature invariants in general relativity and algebraically measures the amount of curvature of spacetime, despite the applied coordinate system [17–19]

$$K = \sum_{\gamma=0}^{3}\sum_{\beta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} R^{\gamma\beta\mu\nu}R_{\gamma\beta\mu\nu}. \quad (27)$$

Thus, the quadratic polynomial invariant is composed of the sum of the squares of Riemann tensor components, $R^{\alpha\beta\mu\nu}$ and $R_{\alpha\beta\mu\nu}$. Eq. (27) shows that K requires horrendous algebraic derivations.

For $g_{\alpha\beta}$ and from Eq. (27) and the affine connections, the Kretschmann scalar is derived as

$$K = +\frac{1}{8r^8}\left\{\left(8Mr-\frac{3Q^2}{\pi\varepsilon}\right)^2+\frac{2}{\pi^2\varepsilon^2}\left(Q^2-8\pi\varepsilon rM\right)^2+\frac{2}{\varepsilon^2}\left(1+\frac{3}{\pi^2}\right)\left(Q^2-4\pi\varepsilon rM\right)^2+\right.$$
$$\left.+\frac{\left[3Q^4+4\pi\varepsilon r\left(Q^2[3r-8M]+8M\varepsilon r[2M-r]\right)\right]^2}{\varepsilon^2\left[Q^2+4\pi\varepsilon r(r-2M)\right]^2}\right\}. \quad (28)$$

With $\tilde{g}_{\alpha\beta}$ and the quantized linear affine connection, Eq. (13), the corresponding Kretschmann scalar, Eq. (27), can be straightforwardly quantized as

$$\tilde{K}=\frac{1}{256}\left\{\frac{16C}{\pi^2\varepsilon^2 r^8}\left(Q^2-4\pi\varepsilon rM\right)^2\left(1+\sin^4\theta\right)+\frac{256C}{r^4}\left[\frac{4r+8rM-\dfrac{Q^2}{\pi\varepsilon}}{4r^2}+\cot\theta+F_{,\gamma}^2\right]^2+\right.$$

$$+256C\Delta^2(r)\left[\frac{1}{r}+\cot\theta+\frac{F_{,\gamma}^2}{C}+\frac{Q^2-4\pi\varepsilon rM}{r^2\left[Q^2+4\pi\varepsilon r(r-2M)\right]}\right]^2+$$

$$+\frac{4096\pi^2\varepsilon^2 r^4 C}{\left[Q^2+4\pi\varepsilon r(r-2M)\right]^2}\left[\frac{1}{r}+\cot\theta+\frac{F_{,\gamma}^2}{C}-\frac{(Q^2-4\pi\varepsilon rM)\left(Q^2+4\pi\varepsilon r[r-2M]\right)}{16\pi\varepsilon^2 r^6}\right]^2+$$

$$+\frac{16}{\pi^2\varepsilon^2 r^6 C}\left[C\left(Q^2+4\pi\varepsilon\left[r^2+r-M-2rM\right]\right)-2\pi\varepsilon rF_{,\gamma}^2(1-2M+2r)\right]^2+$$

$$+\frac{16}{\varepsilon^2 r^8\left(Q^2+4\pi\varepsilon r[r-2M]\right)^2}\left[C\left(Q^4+4\pi\varepsilon rQ^2[r-3M]-16\pi\varepsilon^2 r^2\left(r^3+\pi M[r-2M]\right)\right)-8\pi\varepsilon^2 r^6 F_{,\gamma}^2\right]^2+$$

$$+\frac{Q^2+4\pi\varepsilon[r-2M]}{\pi^3\varepsilon^3 r^{10}}\left[-2CQ^2[r-3]+8\pi\varepsilon rCM[r-2]+r^2F_{,\gamma}^2\left(Q^2+4\pi\varepsilon r[r-2M]\right)\right]^2+$$

$$+\frac{16}{\pi^2\varepsilon^2 r^{12}C}\left[r^2C\left(Q^2-8\pi\varepsilon rM+4\pi\varepsilon r\csc^2\theta\right)+2\pi\varepsilon F_{,\gamma}^2\left(r^4\csc^2\theta+2\cot\theta\left[r^4-\csc^4\theta\right]\right)\right]^2+$$

$$+\frac{8\left(Q^2-4\pi\varepsilon rM\right)}{\pi^2\varepsilon^2 r^8}\left[2CQ^2[r-1]+8\pi\varepsilon rC\left(3M-2r[1+M]+r^2\right)+r^2F_{,\gamma}^2\left(Q^2+4\pi\varepsilon r[r-2M]\right)\right]+$$

$$+\frac{16\left(Q^2-4\pi\varepsilon rM\right)}{\pi^2\varepsilon^2 r^8}\left[CQ^2[1+r]-4\pi\varepsilon rC\left[M+r^2\right]-2\pi\varepsilon r^2F_{,\gamma}^2\csc^2\theta\right]+$$

$$+\frac{64C\Delta(r)}{\pi\varepsilon r^4\left(Q^2+4\pi\varepsilon r[r-2M]\right)^3}\left[2\frac{r}{C}F_{,\gamma}^2\left(Q^4+4\pi\varepsilon rQ^2[r-3M]+4\pi^2\varepsilon^2 r^2\left[8M^2-4rM+r^3\right]\right)\right]+$$

$$+\frac{1}{\varepsilon\left(Q^2+4\pi\varepsilon r[r-2M]\right)}\left[\pi\varepsilon^2\left(-Q^2+4\pi\varepsilon r[r-2M]\right)\left(5Q^4+12\pi\varepsilon rQ^2[r-4M]-32\pi^2\varepsilon^2 r^2M[r-3M]\right)+\right.$$

$$+8\pi\varepsilon^2\left(Q^6+\pi\varepsilon rQ^4[7r-18M]-8\pi^2\varepsilon^3r^3M\left[r^2(2+r+2\pi)+4M^2(2+3\pi)-2rM(4+5\pi)\right]\right)+$$

$$\left.\left.+2\pi\varepsilon^2r^2Q^2\left(r^2[r+6\pi]+M^2[4+46\pi]-2M[r+17\pi r]\right)\right]^2\right\}. \quad (29)$$

The numerical analysis of K, Eq. (28), and $\tilde{K}$, Eq. (29), shall be presented in section III B.

III. Numerical Results and Discussions

To draw qualitative conclusions from the analytical expressions for $d\Theta/d\tau$, Eq. (22), and $d\tilde{\Theta}/d\tau$, Eq. (26), their approximate numerical evolutions at varying C, the generic conformal quantization factor, and r, the radial distance from the singularity, shall be discussed in section III A.

A. Timelike Geodesic Congruence with $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$

For a numerical study of the timelike geodesic congruence of Reissner-Nordström black hole, we suggest approximate values for the quantities $Q=0.5,\ \ M=0.5,\ \ F_{,\gamma}^2=0.5,$ and $\varepsilon=0.01$. They are just *ad hoc* inputs. In absence of any other alternative, such a rough approximation seems to be unavoidable but allows for qualitative conclusions to be made, despite the mathematical constrains.

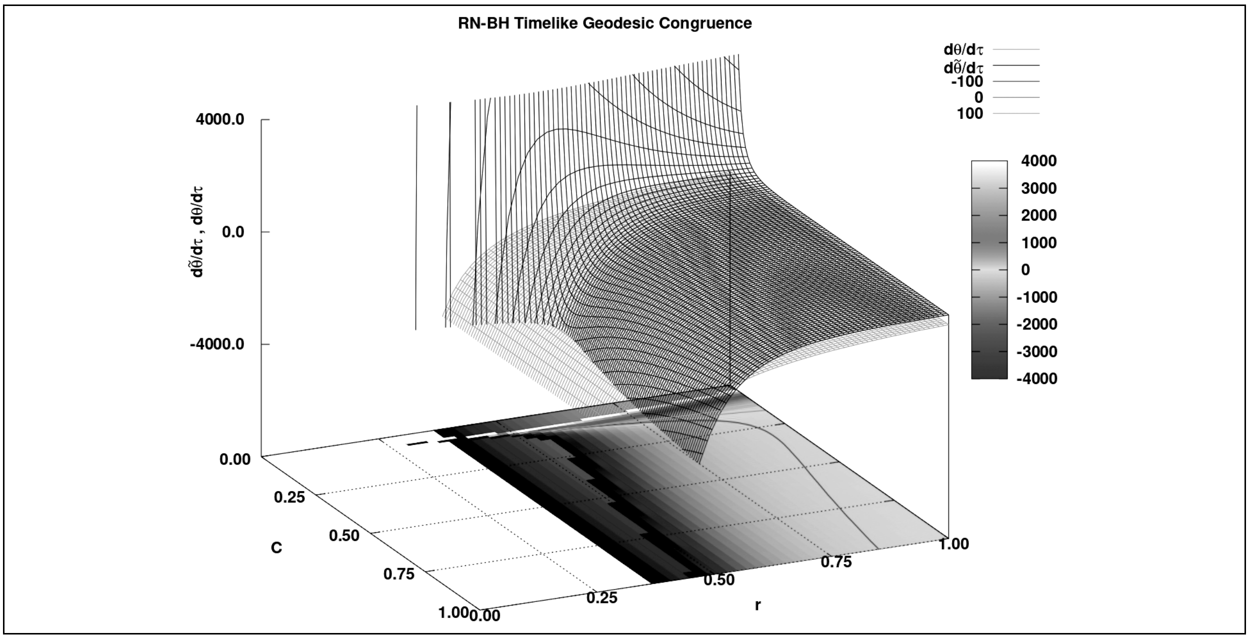

Fig. 1. Top panel: the dependence of the timelike geodesic congruence on the radial distance r and the quantization imposed on the fundamental tensor C. The results of $d\Theta/d\tau$ for $g_{\alpha\beta}$ are confronted with that of $d\tilde{\Theta}/d\tau$ for $\tilde{g}_{\alpha\beta}$. The bottom panel illustrates the same as in the top panel, but here the timelike geodesic congruence, which exclusively emerges by the quantization, namely $d\tilde{\Theta}/d\tau - d\Theta/d\tau$

Fig. 1. Верхняя панель: зависимость временной геодезической конгруэнтности от радиального расстоянияи r и квантование, наложенное на фундаментальный тензор C. Результаты $d\Theta/d\tau$ для $g_{\alpha\beta}$ сталкиваются с тем, что из $d\tilde{\Theta}/d\tau$ для $\tilde{g}_{\alpha\beta}$. Нижняя панель иллюстрирует то же, что и в верхней панели, но здесь временная геодезическая конгруэнтность, которая возникает исключительно при квантовании, а именно $d\tilde{\Theta}/d\tau - d\Theta/d\tau$

The top panel of Fig. 1 depicts the dependence of the timelike geodesic congruence on r, the radial distance, and C, the mean value of the quantization imposed on the fundamental tensor. We compare between the results for $g_{\alpha\beta}$, Eq. (22) (bottom lattice) and $\tilde{g}_{\alpha\beta}$, Eq. (26) (top lattice). The dependence of $d\Theta/d\tau$ on r is expectedly monotonic. In this regard, we keep in mind that such a phenomenological observation likely highly depends on the set of parameters suggested in the present calculations. Furthermore, we observe that as the radial distance r approaches singularity, $d\Theta/d\tau$ rapidly decreases. The remarkable feature we refer to is the presence of negative curvature almost everywhere [19]. For $\tilde{g}_{\alpha\beta}$, the resulting $d\tilde{\Theta}/d\tau$ has a remarkable dependence on r, that the values of $d\tilde{\Theta}/d\tau$ switch between positive and negative signs. The dependence of $d\tilde{\Theta} / d\tau$ on C is also phenomenal. At small C, the numerical values of $d\tilde{\Theta}/d\tau$ approach positive and negative infinity. Obviously, this refers to huge trajectory congruence, which is generated from the velocity fields. We also find that $d\Theta/d\tau$ and $d\tilde{\Theta}/d\tau$ are mostly distinguishable in the $(r-C)$ space. Also, we find that as C approaches unity, $d\tilde{\Theta}/d\tau$ comes closer to $d\Theta/d\tau$. We remark that even at $C=1$, both quantities are not fully identical. A reasoning can be realized from the quantization procedure which was reviewed in Section I. This is much more diverse than just the conformal transformation in Eq. (1).

The additional contributions to $d\Theta/d\tau$ due to the proposed quantization are depicted in the bottom panel of Fig. 1, namely $d\tilde{\Theta}/d\tau - d\Theta/d\tau$. It is obvious that the quantization, whose proposed conformal transformation of $g_{\alpha\beta}$ is just one potion, comes up with a huge timelike geodesic congruence. Large nonmonotonic timelike geodesic congruence occurs, at varying r and C. Relative to $d\tilde{\Theta}/d\tau$, $d\Theta/d\tau$ is very small. This would mean that the GR at the quantum scale seems to reveal rich spacetime-texture which is obviously veiled at the classical scale. A quantitative estimation of timelike geodesic congruence evolution shall be derived elsewhere, especially when the mathematical challenges, Section II, are overcome.

The observation of large timelike geodesic congruence brings up the question whether the corresponding curvature is real or merely prompts as an artifact in some coordinate systems. The section that follows suggests an answer to this question. An assessment of essentiality of the curvature is provided by the invariant scalars of general relativity.

B. Kretschmann Scalar with $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$

It is obvious that Eq. (27) is also valid for the quantized fundamental tensor, Eq. (1), so that

$$\tilde{K} = \sum_{\gamma=0}^{3}\sum_{\beta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\tilde{R}^{\gamma\beta\mu\nu}\tilde{R}_{\gamma\beta\mu\nu}. \tag{30}$$

To summarize the importance of the Kretschmann scalar, we emphasize that i) it gives an algebraic description of the spacetime curvature and singularity, and ii) it does not depend on the coordinate system. This allows to reveal the real nature of the curvature and singularity, especially whether they are intrinsic or removable artifact. The artifact curvature and singularity, i.e., inessential and extrinsic curvatures, apparently emerge in some coordinate systems and expectedly disappear in others. They can be eliminated in some coordinate transformations. On the other hand, the set of the coordinate transformations we know of so far are not forming a closed set, so that the assessment which is merely based on the coordinate transformation would not be capable of distinguishing real from artifact curvature and singularity. The reason is that there might exist other not-yet-discovered coordinate transformations, in which the curvature and singularity turn to be removable artifact.

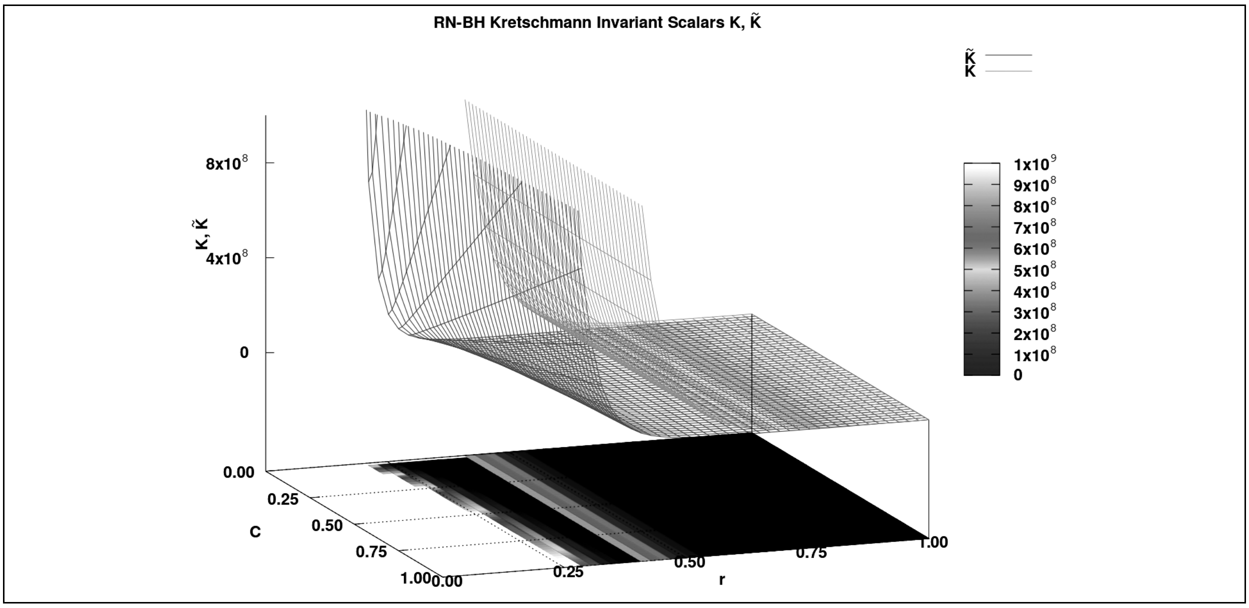

Fig. 2. The Kretschmann scalar is presented as a function of the radial distance to the singularity r and the generic conformal quantization factor C. The results obtained for $g_{\alpha\beta}$ (short lattice) are compared with the results for $\tilde{g}_{\alpha\beta}$ (long lattice)

Рис. 2. Скаляр Кречмана представляется как функция радиального расстояния до сингулярности r и общий конформный коэффициент квантования C. Результаты, полученные для $g_{\alpha\beta}$ (короткая решетка) сравниваются с результатами для $\tilde{g}_{\alpha\beta}$ (длинная решетка)

Figure 2 presents the numerical results of the Kretschmann scalar as a function of r and C. The same set of parameters which was utilized in Fig. 1 is assumed here. We find that K rapidly increases with decreasing r (short lattice). Almost the same observation is found in the dependence of $\tilde{K}$ on r and C (large lattice). That K for conventional $g_{\alpha\beta}$ and $\tilde{K}$ for quantized $\tilde{g}_{\alpha\beta}$ are nonvanishing means that the corresponding curvature and singularity are essential and real in both versions of general relativity. We also find that the quantization seems to locally sharpen the curvature and singularity of the Reissner-Nordström black hole. That is the large values of $\tilde{K}$ seem to emerge at smaller r relative to that of K, i.e., closer to the singularity.

IV. Conclusion

We studied the timelike geodesic congruence of the Reissner-Nordström metric with the conventional $g_{\alpha\beta}$ and quantized fundamental tensor $\tilde{g}_{\alpha\beta}$. While $g_{\alpha\beta}$ characterizes smooth continuous Riemann geometry, $\tilde{g}_{\alpha\beta}$ is imposed on discretized intermittent Riemann geometry. We found that the evolution of the geodesic congruence expansion is nonvanishing in the gravitational field of the charged, non-rotating, spherically symmetric, and massive Reissner-Nordström black hole. Concretely, decreasing the radial distance from singularity manifests extremely large evolution of the geodesic congruence expansion. Such a monotonic dependence is largely dictated by the approximate values assigned to the parameter M (mass), Q (electric charge) and ε (vacuum permeability or electric and magnetic constants). The proposed quantization obviously enhances the evolution of the geodesic congruence expansion and imposes a remarkable dependence on the radial distance r and the generic conformal quantization factor C. Here, the geodesic congruence expansion rate promptly switches between large positive and large negative values, especially at small C.

It is worthy mentioning that the proposed quantization preserves the main assumptions of general relativity. Not only the fundamental tensor, but also the affine connections, geodesic equations, and Riemann curvature tensor all can be fully retrieved, at $C=1$. Therefore, the evolution of the geodesic congruence expansion with quantized fundamental tensor accommodates the results which are to be obtained with the conventional fundamental tensor. This allows to determine the contributions from the proposed quantization. Here, the dependence on the radial distance to the singularity is remarkable. At distances closer to the singularity, the evolution becomes huge and suddenly switches between large positive and large negative values. Comparing to the results of the classical geodesic congruence expansion evolution, the additional quantum contributions reveal that GR seems to have a rich spacetime-structure at the quantum scale which is largely overseen in the classical limit.

To assess whether the emerged curvature and singularity are real and essential, we analyzed the Kretschmann scalar K and found that values of K which are obtained for $g_{\alpha\beta}$ and the values of $\tilde{K}$ which are obtained for $\tilde{g}_{\alpha\beta}$, are finite everywhere. They largely increase while approaching the singularity. Therefore, we conclude that the curvature and singularity obtained with both versions of the fundamental tensor are essential and real. Furthermore, we observe that the quantization locally sharpens the corresponding curvature and singularity. Whether this phenomenological finding refers to a revision to be imposed on the Schwarzschild radius should be analyzed elsewhere.

Last but not least, we conclude that the curvature and singularity of charged, non-rotating, spherically symmetric, and massive Reissner-Nordström black hole are likely accompanied with additional sources of quantum-mechanically originating gravitational sources that might alter the black hole geometry, especially in the relativistic and quantum regime.

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Information about the authors

Abdel Nasser Tawfik – D.Sc, Professor
Azzah Aziz Alshehri – Ph.D., Professor

The article was submitted 21.05.2024
Approved after reviewing 09.07.2024
Accepted for publication 28.08.2024

Научная статья

Черная дыра Рейснера–Нордстрема: кривизна и сингулярность с квантованным фундаментальным тензором

А.Н. Тауфик[1], **А.А. Альшехри**[2]

[1] Университет Будущего в Египте (FUE),
Пятое поселение (11835, Новый Каир, Арабская Республика Египет)
[2] Университет Хафра эль-Батина (Хафар-эль-Батин, 39524, Королевство Саудовская Аравия)
[1] a.tawfik@fue.edu.eg

Аннотация

Постановка проблемы. Выявление природы кривизн и сингулярностей, возникающих при квантовании фундаментального тензора согласно предложенной модели, требует аналитического получения и численного анализа времениподобной геодезической конгруэнции для метрики Рейснера-Нордстрема. Выясняется, что при дальнейшей эволюции расширение геодезической конгруэнции нигде не обращается в ноль. Кроме того, по мере уменьшения радиального расстояния от сингулярности эволюция расширения геодезической конгруэнции становится чрезвычайно большой. Предложенное квантование, по-видимому, приводит к значительным изменениям профиля эволюции расширения геодезической конгруэнтности в сторону усиления.
Цель. Поскольку выявлено, что скаляр Кречмана конечен на всей области определения для обоих вариантов фундаментального тензора, возможно утверждение, что кривизны и сингулярности, вероятно, реальны и существенны (а не являются искажением, привносимым использованием определенных координатных систем).
Результаты. Предложенное квантование, по-видимому, приводит к локальному росту значений кривизны, а следовательно, и сингулярности для заряженной, невращающейся, сферически симметричной массивной черной дыры Рейснера-Нордстрема.
Практическая значимость. Возможно, данное открытие приведет к пересмотру радиуса Шварцшильда или даже всей геометрии черной дыры, особенно в масштабах релятивистской квантовой механики. Можно также заключить, что даже приблизительная качественная оценка дополнительных кривизн указывает на сложную пространственно-временную структуру, которую, очевидно, невозможно выявить при приближении к классической механике.

Ключевые слова

Пространство и космологические сингулярности; инвариантные скаляры в общей теории относительности; дискретное искривленное пространство-время; геометрия Римана-Финслера-Гамильтона; некоммутативная дифференциальная геометрия

АТ благодарит за поддержку Университет будущего в Египте (FUE), Египетский центр теоретической физики (ECTP) и Всемирную лабораторию космологии и физики элементарных частиц (WLCAPP)!

Для цитирования

Тауфик А.Н., Альшехри А.А. Черная дыра Рейснера–Нордстрема: кривизна и сингулярность с квантованным фундаментальным тензором // Нелинейный мир. 2024. Т. 22. № 3. С. 49–61. DOI: https://doi.org/10.18127/j20700970-202403-06 (на английском)

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Информация об авторах

Абдель Насер Тауфик – д.н., профессор
Аззах Азиз Альшехри – к.н., профессор

Статья поступила в редакцию 21.05.2024
Одобрена после рецензирования 09.07.2024
Принята к публикации 28.08.2024