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Applications of optimal control to general relativity

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Abstract

Two classes of applications of optimal control to problems in General Relativity are reviewed. The first class includes direct applications where each problem has beforehand a specific objective function. Two examples of this class are reviewed, where optimal control leads directly to the optimal solutions. Results show that optimal control is more powerful than classical variational calculus. The second class includes innovative applications where problems in General Relativity may be approached by introducing appropriate objective functions. Three examples of this class are reviewed, where an optimal inflationary universe, an optimal cosmological model and an optimal stellar model are, respectively, constructed. Results show that optimal control adds physical significance to solutions of such problems.

Keywords

General relativity, stellar models, cosmological models, optimal control

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A brief version in Russian is given at the end of the article

I. Introduction

General Relativity (GR) is Einstein's theory of gravitation. In GR, the interaction between the mass-energy distribution and the spacetime geometry is described by Einstein's field equations. The theory of optimal control deals with optimal control problems, which require control functions to be optimal with respect to given criteria. The theory has many interesting applications in various fields of engineering, economics, management, and biology. However, quite a few applications have been made to problems in GR. These applications may be classified into two classes. The first class includes direct applications where each problem has beforehand a specific objective function. The second class includes innovative applications where problems in relativistic astrophysics and cosmology may be approached by introducing appropriate objective functions.

In this paper we present a review of two direct applications. Results show that optimal control is more powerful than classical variational calculus. Then, we present a review of three innovative applications. Results show that optimal control adds physical significance to solutions of some problems in GR.

The paper is organized as follows. Section II gives a brief outline of optimal control problems. Section III gives a brief outline of relativistic stellar and cosmological models. In section IV, two direct applications of optimal control to problems in GR are reviewed. Section V introduces innovative applications and gives a review of three applications, where an optimal inflationary universe, an optimal cosmological model and an optimal stellar model are, respectively, constructed. In the concluding section VI, extension to other problems is suggested, and difficulties in the formulation of optimal control problems in GR are indicated.

II. Optimal control problems

An optimal control system is defined in terms of an n-dimensional state vector $x(t)$ and an m-dimensional control vector $u(t)$. An optimal control problem [1–3] is to determine the control vector function u which maximizes the objective function

$$J = \int_{t_0}^{t_1} L(x,u,t)dt + F(x_1,t_1), \tag{1}$$

where the system evolution equation is

$$\dot{x} = f(x,u,t), \tag{2}$$

subject to the constraints

$$x(t_0) = x_0 \ , \quad \psi(x_1,t_1) = 0, \qquad u \in \Omega. \tag{3}$$

The functions L and F are scalars. The terminal constraint function ψ is an s-dimensional vector. The set Ω is the set of admissible controls. The initial time t_0 is given explicitly but the terminal time t_1 may be unspecified.

The Hamiltonian for this problem is

$$H(x,u,\lambda,t) = L(x,u,t) + \ < \lambda, f(x,u,t) > \tag{4}$$

where λ denotes an *n*-dimensional vector of costate functions.

According to Pontryagin's maximum principle, the following necessary conditions hold along an optimal trajectory.

$$\bar{u} = \arg\min H(\bar{x},u,\lambda,t), \tag{5}$$

$$\dot{\lambda} = -H_{,x}(\bar{x},\bar{u},\lambda,t), \tag{6}$$

$$\lambda(t_1) = \ F_{,x_1} + < \psi_{,x_1}, \nu >, \tag{7}$$

$$H(t_1) = -F_{,t_1} - \ < \psi_{,t_1}, \nu >, \tag{8}$$

where a comma denotes a partial derivative, $\bar{x}$ and $\bar{u}$ denote the candidate state and control vectors respectively, and ν is an *s*-dimensional vector of constant Lagrange multipliers associated with ψ.

The terminal conditions (7) and (8) result from the transversality condition

$$[(F_{,x_1} - \lambda(t_1))\delta x + (H(t_1) + F_{,t_1})\delta t]_{\psi(x_1,t_1)} = 0. \tag{9}$$

If some initial parameters are not specified similar conditions may be obtained at the initial time t_0.

The optimal control problem in the above form is referred to as a problem of Bolza. If $F(x_1,t_1) = 0$ then the problem is a problem of Lagrange, while if $L(x,u,t) = 0$ then it is a problem of Mayer. The three problems are equivalent: each of them can be converted to the other by suitably defining the state functions.

III. Stellar and cosmological models

One of the main applications of GR is the construction of stellar models [4]. The simplest such model is an isolated static perfect fluid sphere, of coordinate radius R and total mass *M*. The vacuum outside the sphere has the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad r \geq R. \tag{10}$$

In the interior of the sphere, the spacetime is described by the static spherically symmetric metric

$$ds^2 = e^{2\nu} dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad r \leq R, \tag{11}$$

where ν and m are functions of the radial coordinate r. The energy-momentum tensor for a perfect fluid is given by

$$T^{ab} = (\mu + p)U^a U^b - pg^{ab}, \tag{12}$$

where μ and p are, respectively, the fluid density and pressure, and U^a is the four-velocity of the fluid. The field equations reduce to

$$m' = 4\pi r^2 \mu, \tag{13}$$

$$p' = -\frac{(\mu + p)(m + 4\pi r^2 p)}{r(r - 2m)}, \tag{14}$$

$$\nu' = -\frac{2p'}{(\mu + p)}, \tag{15}$$

where a prime denotes differentiation with respect to r.

Another important application of GR is the construction of cosmological models. The simplest such model is a homogeneous and isotropic spacetime, filled with perfect fluid, with the Friedmann-Robertson-Walker (FRW) metric [4]

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{16}$$

where a is a function of the cosmic time t, and $k = -1, 0, +1$ corresponds to open, flat and closed universe respectively. The function $a(t)$ is the *scale factor*.

Using equation (12), Einstein's field equations, with a *cosmological constant* Λ, may be reduced to

$$3H^2 + \frac{3k}{a^2} = 8\pi\mu + \Lambda, \tag{17}$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi p + \Lambda, \tag{18}$$

where a dot denotes differentiation with respect to t, and H is the Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{19}$$

IV. Direct applications

Some problems in GR seek the maximum (minimum) of a functional. If the problem lies within the calculus of variations, the solution may be obtained by solving the Euler-Lagrange equations. If the problem has additional variables involved other than the state variables, the solution may be obtained by direct application of optimal control. However, in dealing with such more general problems, researchers tend to use classical variational calculus. We review two direct applications of optimal control to problems in relativistic stellar models.

IV.1. The maximum mass of a neutron star

The first problem that has been solved by direct application of optimal control was the calculation of the maximum mass of a neutron star [5], which depends on the equation of state (EOS). However, EOS can be predicted only at certain density ranges. For neutron matter with higher densities, EOS cannot be predicted. By imposing only some physical constraints at higher densities, [6] suggested that the mass of a stable neutron star becomes maximum for the stiffest possible EOS.

By using optimal control, [5] successfully proved this suggestion; namely that, in the regions where it is uncertain, EOS that gives the maximum mass is $p=\mu$.

With a few simplifications, their work can be summarized as follows. Consider a neutron star with field equations (13-15) for a static perfect fluid sphere. Let the star be composed of two regions:

(a) Region A: an inner core with higher densities, where

$$0 \le r \le r_0, \quad \mu_c \ge \mu \ge \mu_0, \quad p_c \ge p \ge p_0. \tag{20}$$

(b) Region B: an outer envelope with given EOS $p = p(\mu)$, where

$$r_0 \le r \le R, \quad \mu_0 \ge \mu \ge \mu_R, \quad p_0 \ge p \ge 0. \tag{21}$$

The mass contained in B depends not only on the given EOS, but also on the values at the boundary $r=r_0$, and hence it is denoted $M_0(\mu_0, r_0, m_0)$. For the dynamical system, take μ as the independent variable, the functions p, r, m as state functions, and $u=\dot{p}$ as the control function. Then, the optimal control problem is formulated as follows. Determine the control function $u(\mu)$ which maximizes the objective function

$$M = \int_{\mu_c}^{\mu_0} 4\pi r^2 \, \dot{r} \, d\mu + M_0(\mu_0, r_0, m_0), \tag{22}$$

subject to the system equations and constraints

$$\dot{p} = u, \quad p(\mu_c) = p_c, \tag{23}$$

$$\dot{r} = -\frac{r(r-m)u}{(p+\mu)\left(m+4\pi r^3 p\right)}, \quad r(\mu_c) = 0, \tag{24}$$

$$\dot{m} = -\frac{4\pi\mu r^3 (r-m)u}{(p+\mu)\left(m+4\pi r^3 p\right)}, \quad m(\mu_c) = 0, \tag{25}$$

$$0 \le u \le 1, \tag{26}$$

where a dot denotes differentiation with respect to μ.

Application of Pontryagin's maximum principle leads to the optimal control $u = \dot{p} = 1 \Rightarrow p = \mu$ with possibly a switch to $u = \dot{p} = 0 \Rightarrow p = p_c$. Further calculations result in a maximum mass of about $3.2 M_\Theta$ for $\mu_0 = 4.6 \times 10^{14}$ gcm^{-3}. Details are given in [5].

IV.2. The equation of hydrostatic equilibrium

The first law of thermodynamics relates the number density of baryons n to μ and p by the equation

$$\frac{dn}{d\mu} = -\frac{n}{p+\mu}. \tag{27}$$

The following theorem has been proved [7, 8].

Theorem 1. Among all momentarily static and spherically symmetric configurations of cold, catalyzed matter which contain a specific number of baryons inside a sphere of radius R,

$$N = \int_0^R 4\pi r^2 \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} n(r)\, dr,$$

that configuration which extremizes the mass as sensed from outside,

$$M = \int_0^R 4\pi r^2 \, \mu(r)\, dr,$$

satisfies equation (14), the Tolman-Oppenheimer-Volkoff (TOV) equation of hydrostatic equilibrium

$$p' = -\frac{(p+\mu)\left(m + 4\pi r^3 p\right)}{r(r-m)}, \tag{28}$$

Proofs to the theorem have been given in [7, 8], using classical variational calculus, with each of them filling about two pages. Rather than using classical variational calculus, [9] gives a proof using optimal control. The problem is formulated as an optimal control problem as follows. Determine the control function $n(r)$ which extremizes the objective function

$$M = m(R) = \int_0^R 4\pi r^2 \, \mu(n)\, dr, \tag{29}$$

where the system equations and constraints are

$$m' = 4\pi r^2 \mu(n), \quad m(0) = 0, \tag{30}$$

$$z' = 4\pi r^2 \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} n, \quad z(0) = 0, \quad z(R) = N. \tag{31}$$

Application of Pontryagin's maximum principle led directly to the TOV equation, in a way that is simpler and more elegant than the classical variational calculus. Details are given in [9].

V. Innovative applications

Einstein's interior field equations are potential candidates for optimal control application. It is well known that the interior field equations are under-determined, in the sense that the number of equations is less than the number of variables. To obtain an interior solution, an additional equation has to be introduced. For example, an equation of state is to be specified, or a relation between some variables is assumed so that the problem is simplified. An alternative approach is to formulate the problem as an optimal control model. In such a model, state functions are governed by differential equations that include some unknown control functions. The problem is initially under-determined. But the model also incorporates a functional, of the state and control functions, that is required to be optimal. Then, the conditions of optimality determine the control functions and the problem becomes determined. This approach could lead to new physical interpretation and deeper understanding of the construction of interior field equations and their solutions. Such applications of optimal control to GR may be coined innovative applications. We review three innovative applications, where an optimal inflationary universe, an optimal cosmological model and an optimal stellar model are, respectively, constructed.

V.1. A slow-roll inflationary universe

Inflationary models are based on the effective theory of a self-acting scalar field φ with self-action potential $V(\varphi)$, which interacts minimally with the gravitational field [10]. For such models, with the FRW metric, the field equations, with Λ term, reduce to

$$\dot{H} + 3H^2 + \frac{2k}{a^2} = 8\pi V + \Lambda, \tag{32}$$

$$-3\left(\dot{H} + H^2\right) = 8\pi\left(\dot{\varphi}^2 - V\right) - \Lambda, \tag{33}$$

$$\ddot{\varphi}+3H\dot{\varphi}+\frac{dV}{d\varphi}=0. \qquad (34)$$

Any one of equations (32)–(34) may be derived from the other two. Thus, we have a system of two independent equations in three unknowns a, φ and V.

In order to obtain a solution, an additional assumption is needed. For example [11] and [12] assumed forms of $V=V(\varphi)$, [13] assumed forms of $a=a(t)$, and [14] assumed forms of $\varphi=\varphi(t)$.

Rather than using an arbitrary assumption, Zhuravlev et al [15] suggested to construct a model of slow-roll inflationary universe with minimum change of the scalar field.

A slow-roll regime requires a slow change in the field φ. Thus, the evolution of the universe must be such that on any finite time interval $[t_1,t_2]$ would be the smallest among all other possible evolutions.

By adding equations (32) and (33), one obtains

$$\dot{\varphi}=\sqrt{\frac{1}{4\pi}}\sqrt{\frac{k}{a^2}-\dot{H}}, \qquad (35)$$

and then

$$\Delta\varphi=\int_{t_1}^{t_2}\dot{\varphi}\,dt=\sqrt{\frac{1}{4\pi}}\int_{t_1}^{t_2}\sqrt{\frac{k}{a^2}-\dot{H}}\,dt. \qquad (36)$$

Zhuravlev et al [15] obtained a solution using calculus of variations. Haggag et al [16] obtained a solution using optimal control as follows. Taking a and H as the state functions, and $u=\dot{H}$ as the control function the optimal control problem is then formulated as follows.

Determine the control function u which minimizes the objective function

$$J=\int_{t_1}^{t_2}\sqrt{\frac{k}{a^2}-u}\;dt, \qquad (37)$$

subject to

$$\dot{a}=aH, \quad a(t_1)=a_1=\text{const}, \qquad (38)$$

$$\dot{H}=u, \quad H(t_1)=H_1=\text{const}. \qquad (39)$$

It should be noted that, given a solution a, H, u of the above problem, the scalar field φ is determined by integration of equation (35), and the potential V is determined as an output of the system by

$$8\pi V=\frac{2k}{a^2}+3H^2+u-\Lambda, \qquad (40)$$

Application of Pontryagin's maximum principle leads to

$$2a^4\left(k-a^2u\right)\ddot{u}+3a^6\dot{u}^2+12ka^2\dot{a}^2u+4k^2a\ddot{a}+4ka^3\left(3\dot{a}\dot{u}-u\ddot{a}\right)+8k^3+8ka^4u^2-16k^2a^2u=0. \qquad (41)$$

For a flat universe, $k=0$, the optimal solution is given by

$$u(t)=\frac{-1}{\left(a_0t+b_0\right)^2}, \quad a_0,b_0\ \text{constants}, \qquad (42)$$

$$H(t)=\frac{1}{a_0}\frac{1}{\left(a_0t+b_0\right)}+c_0, \qquad (43)$$

$$a(t) = a_2 \left(a_0 t + b_0\right)^{\frac{1}{a_0^2}} e^{c_0 t}, \tag{44}$$

where

$$c_0 = H_1 - \left[a_0 \left(a_0 t_1 + b_0\right)\right]^{-1}, \quad a_2 = a_1 \left(a_0 t_1 + b_0\right)^{-\frac{1}{a_0^2}} e^{-c_0 t_1}.$$

Then one obtains

$$\varphi(t) = \sqrt{\frac{1}{4\pi}} \frac{1}{a_0} \ln\left(a_0 t + b_0\right) + \varphi_0, \quad \varphi_0 = \text{const.} \tag{45}$$

Finally, equations (42) and (40) give the potential

$$V(t) = \frac{1}{8\pi}\left[-\Lambda + 3c_0^2 + \frac{-1}{\left(a_0 t + b_0\right)^2}\left(\frac{3}{a_0^2} - 1\right) + \frac{6}{a_0}\left(\frac{c_0}{a_0 t + b_0}\right)\right]. \tag{46}$$

The potential V, as a function of φ, is then given by

$$V(\varphi) = \frac{1}{8\pi}\left[-\Lambda + 3c_0^2 + \left(\frac{3}{a_0^2} - 1\right) e^{-a_0\sqrt{16\pi}(\varphi - \varphi_0)} + \frac{6}{a_0} c_0 e^{-a_0\sqrt{4\pi}(\varphi-\varphi_0)}\right]. \tag{47}$$

This solution with different constants, has been obtained in [15] using calculus of variations.

V.2. A closed universe with maximum life-time

Many cosmological models consider Einstein's field equations with a varying cosmological constant and a linear equation of state

$$p = \alpha\mu, \quad \alpha = \text{const}, \quad 0 \le \alpha \le 1. \tag{48}$$

Then, equations (17) and (18) give

$$2\ddot{H} + 6H\dot{H}(1+\alpha) - \frac{2k}{a^2} H(1+3\alpha) = \dot{\Lambda}(1+\alpha). \tag{49}$$

Such models have two equations (19) and (49) in three unknowns $a(t)$, $H(t)$ and $\Lambda(t)$. In order to obtain solutions, authors assumed various forms of Λ, for example [17–25]. Rather than using an arbitrary form of Λ, optimal control has successfully been used [26] to construct a model where Λ is determined by the requirement of optimality.

For a closed universe, $(k = +1)$, the scale factor a increases from initial size until $\dot{a} = 0$ then contracts. A closed universe has a finite life-time. It is interesting to design a closed universe with maximum life-time. Equations (19) and (49), with $k = +1$, take the form

$$\dot{a} = aH, \tag{50}$$

$$\dot{\Lambda} = \frac{2}{(1+\alpha)} \ddot{H} + 6H\dot{H} - \frac{2}{a^2}\left(\frac{1+3\alpha}{1+\alpha}\right) H, \tag{51}$$

which is a system of two equations in three unknowns $a(t)$, $H(t)$ and $\Lambda(t)$. Using equations (18), (19) and (48) one obtains

$$\Lambda = \frac{1}{1+\alpha}\left[2\dot{H} + 3H^2(1+\alpha) + \frac{1}{a^2}(1+3\alpha)\right], \tag{52}$$

$$\mu = \frac{1}{8\pi}\left[3H^2 + \frac{3}{a^2} - \Lambda\right], \quad p = \alpha\mu. \tag{53}$$

For the dynamical system (50) and (51), the functions a, H, $w = \dot{H}$, Λ are taken as state functions, and $u = \dot{w}$ as the control function.

Further, boundary conditions need to be specified. In order to avoid the Big Bang singularity, the initial time, $t = t_i = 0$, is taken shortly after the Big Bang moment, when the size of the universe becomes little greater than zero, namely $a(0) = a_0 > 0$. Assuming $H(0) = 0$, an expanding universe requires $\dot{H}(0) = A > 0$. On the other hand, the terminal time $t = t_f = T$ is taken when expansion stops, and the size of the universe reaches its maximum, namely $H(T) = 0$ and $\dot{H}(T) < 0$.

Then, the optimal control problem is to determine the control function $u(t)$ which maximizes the objective function

$$J = \int_0^T dt, \tag{54}$$

subject to

$$\dot{a} = aH, \quad a(0) = a_0, \tag{55}$$

$$\dot{H} = w, \quad H(0) = H(T) = 0, \tag{56}$$

$$\dot{w} = u, \quad w(0) = A > 0, \tag{57}$$

$$\dot{\Lambda} = 6Hw - \frac{2}{a^2}\left(\frac{1+3\alpha}{1+\alpha}\right)H + \frac{2}{(1+\alpha)}u, \quad \Lambda(0) = \Lambda_0, \tag{58}$$

$$|u| \le 1. \tag{59}$$

Application of Pontryagin's maximum principle leads to the optimal solution

$$u = -1, \tag{60}$$

$$H = -\frac{1}{2}t^2 + At, \tag{61}$$

$$a(t) = a_0 \exp\left[\frac{-1}{6}t^2(t - 3A)\right]. \tag{62}$$

The maximum life-time is

$$2T = 4A. \tag{63}$$

The maximum value of the scale factor is

$$a_{\max} = a(T) = a_0 \exp\left[\frac{2}{3}A^3\right]. \tag{64}$$

This universe expands to the above maximum size at $t = T$, and then contracts to

$$a_f = a(2T) = a_0 \exp\left[-\frac{8}{3}A^3\right]. \tag{65}$$

V.3. A stellar model with minimum rest mass

The simplest stellar models are described by equations (10)–(15). Such models have three field equations (13)–(15) in four unknowns v, m, μ and p. To obtain a solution, an additional ansatz needs to be prescribed. Haggag [27] has successfully used optimal control to construct stellar models with minimum rest mass in three cases.

Case 1. Configurations with piece-wise continuous density

Taking $m(r)$ as the state function and $\mu(r)$ as the control function, the optimal control problem is formulated as follows. Determine the control μ which minimizes the objective function

$$M_0 = \int_0^R 4\pi\mu r^2 \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr,$$

subject to the system equation and constraints

$$m' = 4\pi r^2 \mu, \quad m(0) = 0, \quad m(R) = M,$$
$$\mu \geq 0 \quad \text{piece} - \text{wise continuous on } [0, R].$$

The Hamiltonian takes the form

$$H = 4\pi\mu r^2 \left[\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} + \lambda_m\right],$$

Where λ_m is the costate function associated with m, and should satisfy the adjoint equation

$$\lambda_m' = -4\pi\mu r \left(1 - \frac{2m}{r}\right)^{-\frac{3}{2}},$$

Noting that H is linear in μ a singular control [18] satisfying $H, \mu = 0$ on an interval in $[0, R]$ is found as a Dirac delta function $m = \sigma\delta(r - R)$, with $4\pi\sigma R^2 = M$. Thus, the optimal solution is the spherical shell.

Since M is constant, the binding energy, $M_0 - M$, is minimum. Bizon et al [28] obtained the same result. However, in their work they started with that result as a conjecture, then used some clever, non-systematic, steps to show that the binding energy of a spherical shell is less than that of other configurations. The proof in [27] uses no prior conjecture. It uses systematic steps of optimal control to, more elegantly, derive the result.

Case 2. Configurations with piece-wise continuous density gradient

Taking $m(r)$ and $\mu(r)$ as the state functions, and $x := \mu'(r)$ as the control function, the optimal control problem is formulated as follows. Determine the control x which minimizes M_0 subject to the system equations and constraints

$$m' = 4\pi r^2 \mu, \quad m(0) = 0, \quad m(R) = M,$$
$$\mu' = x$$
$$x \leq 0 \quad \text{piece} - \text{wise continuous on } [0, R].$$

The Hamiltonian takes the form

$$H = 4\pi\mu r^2 \left[\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} + \lambda_m\right] + x\lambda_\mu,$$

where λ_μ is the second costate function associated with μ, and should satisfy the adjoint equation

$$\lambda_\mu' = -4\pi r^2 \left[\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} + \lambda_m\right], \ \lambda_\mu(0) = 0, \quad \lambda_\mu(R) = 0.$$

The boundary conditions on λ_μ are implied by the transversality conditions when μ is not specified at either ends.

Noting that H is linear in μ a non-singular control is found where λ_μ, the coefficient of x, is not identically zero. Then, when λ_μ is negative, H will be minimum when x takes its greatest value 0. An exact expression for λ_μ is found, showing that it is indeed negative on $(0, R)$. The optimal solution is

$$m = 4\pi\mu_0 r^3 / 3, \quad \mu = \mu_0 = 3M / 4\pi R^3.$$

Thus, if μ has a piece-wise continuous gradient on [0, R], the optimal solution is the interior Schwarzschild solution. The solution is not, of course, new. However, it has now been obtained using a completely new approach. Rather than pre-specifying the equation of state μ = const, it is derived as an optimal solution. This has the advantage of adding a new physical significance: the solution has the minimum rest mass.

As a side benefit, [27] noted that [29] had given expressions for the rest mass for Tolman VII solution [30] and Buchdahl solution [31], both being less than the rest mass of the interior Schwarzschild solution. It is then concluded that those expressions are incorrect.

Case 3. Configurations with the speed of sound not exceeding that of light

In this case, $m(r)$, $\mu(r)$ and $p(r)$ are taken as the state functions, and $z := d\mu/dp \geq 1$ as the control function. It is convenient to define

$$K := -p' = \frac{(\mu + p)(m + 4\pi r^2 p)}{r(r - 2m)} > 0.$$

Then, the optimal control problem is formulated as follows. Determine the control z which minimizes M_0 subject to the system equations and constraints

$$m' = 4\pi r^2 \mu, \quad m(0) = 0, \quad m(R) = M,$$
$$\mu' = -Kz$$
$$p' = -K, \quad p(R) = 0,$$
$$z \geq 1 \quad \text{piece-wise continuous on } [0, R].$$

The Hamiltonian takes the form

$$H = 4\pi\mu r^2 \left[\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} + \lambda_m \right] - Kz\lambda_\mu - K\lambda_p,$$

where λ_p is the third costate function associated with p, and should satisfy $\lambda_p(0) = 0$ since p is not specified at the center.

With H linear in z, a non-singular control is found where $-K\lambda_\mu$, the coefficient of z, is not identically zero. Then, with $K > 0$, when λ_μ is negative, H will be minimum when z takes its least value 1. An approximate solution for λ_μ is found, showing that it is indeed negative on (0, R). The optimal solution has the linear equation of state $\mu = p + \mu_b$, where $\mu_b > 0$ is the density at the boundary $r = R$, determined by M and R.

The linear equation of state was first suggested by Buchdahl and Land [32] to replace the unphysical constant density, as representing a relativistic incompressible fluid. They argued that, in a relativistic situation, incompressibility requires that the speed of sound should be the largest possible, that is the speed of light. Earlier, Eddington [33] suggested $\mu - 3p$ = const. Later, Cooperstock and Sarracino [34] suggested $\mu_{\text{proper}} := \mu / |g_{tt}|^{1/2}$ = const. The results here support the suggestion of the linear equation of state. It is induced by the same physical motivation as the constant density, but with the additional relativistic constraint.

VI. Conclusion

We have given a review of direct applications of optimal control to two problems in relativistic astrophysics. Each problem, with a specific objective function beforehand, has been formulated and solved as optimal control problem.

The first problem determines the configuration of a neutron star with maximum mass. The clever formulation of this problem yielded a typical optimal control problem that had been solved using optimal control methods.

The second problem gives a proof of a variational principle that is brief and straightforward, compared to proofs using classical variational calculus. This advantage should be expected. Pontryagin's maximum principle already used variational calculus to derive the optimality conditions for a general problem form. Once the problem at hand is cast into that form, the optimality conditions can directly be applied. Variational principles are common in branches of physics, and could be candidates for optimal control application.

For stellar models, Fujisawa et al [35] established a bound on the maximum M/R ratio, where a numerical solution was given. The problem was initially modeled as an optimal control problem. However, it was solved using variational calculus that required long derivations. It would be interesting to solve that problem using optimal control methods.

We have also given a review of innovative applications of optimal control to three problems in relativistic astrophysics and cosmology. These problems are different; no objective function is initially given. However, each has, indeed, been modeled as an optimal control problem. Rather than using adhoc assumptions, an objective function is sought to be optimal. Then, the well-established techniques of optimal control would result in the solution, with the advantage of adding a new physical significance.

This approach may be extended to other problems with different configurations such as rotating stars, binary systems, inhomogeneous and/or rotating universes. It may also be extended to problems with more realistic matter having, for example, anisotropic pressure and/or viscosity.

However, at constructing optimal control problems, specifying state and control functions is, sometimes, not a simple task, and should take utmost care. A serious challenge is to specify some optimality criteria that should be physically relevant and should also yield a mathematically solvable problem. It would take a lot of hard work but could, at the end, lead to interesting and powerful results.

It is worth-noting that a different application of optimal control to GR has recently been published. Ansel [36] presented a path integral formalism, based on the framework of optimal control theory, with a new Lagrangian different from the Einstein-Hilbert Lagrangian [37]. Einstein's field equations are then recovered exactly with variations of the new action functional.

The reviewed problems demonstrate the advantages of adopting such an approach, and motivate further applications of optimal control to problems in GR.

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Научная статья

Применение оптимального управления к общей теории относительности

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Аннотация

Постановка проблемы. Рассматриваются подходы к применению оптимального управления в задачах общей теории относительности
В первом подходе для каждой задачи определена целевая функция. Рассмотрены два примера решений таких задач непосредственно методами оптимального управления.
Во втором подходе в задачах общей теории относительности можно ввести соответствующие целевые функции. Рассмотрены три примера решения таких задач, в рамках которых строятся, соответственно, оптимальная инфляционная вселенная, оптимальные космологическая и звездная модели.
Цель. Сравнить два подхода к применению оптимального управления в задачах общей теории относительности.
Результаты. Полученные результаты показывают, что классическое вариационное исчисление в рассмотренных примерах уступает оптимальному управлению. При оптимальном управлении решение задач общей теории относительности получает дополнительный физический смысл.
Практическая значимость. В задачах общей теории относительности, ранее решавшихся иными способами, можно ввести соответствующие целевые функции. В рамках рассматриваемых примеров строятся, соответственно, оптимальная инфляционная вселенная, оптимальная космологическая модель и оптимальная звездная модели. Данные результаты показывают, что решения подобных задач получают дополнительный физический смысл благодаря применению оптимального управления.

Ключевые слова

Общая теория относительности, звездные модели, космологические модели, оптимальное управление

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